Geometric vs non-geometric rough paths

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A rough outline

1 - What did Peter say? (Geometric rough paths)

2 - What did Max say? (Non-geometric rough paths)

3 - Show that non-geometric rough paths are actually geometric.

4 - What does this mean for SDEs? (actually RDEs)

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The problem

We are interested in equations of the form

$$dY_t = \sum_i f_i(Y_t) dX_t^i ,$$

where $X : [0, T] \to V$ is path with *some* Hölder exponent $\gamma \in (0, 1)$, $Y : [0, T] \to U$ and $f_i : U \to U$ are smooth vector fields.

The theory of **rough paths** (Lyons) tells us that we should think of the equation as

$$dY_t = \sum_i f_i(Y_t) d\mathbb{X}_t , \qquad (\dagger)$$

where $\mathbb{X}$ is an object containing X as well as information about the iterated integrals of X. We call $\mathbb{X}$ a **rough path** above X.

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What is a geometric rough path

- $\mathbb{X}$ lives in the tensor product space $\mathbb{X} : [0, T] \to V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes N}$ where N is the largest integer such that $N\gamma \leq 1$.
- $\mathbb{X}$ lives above X in that $\langle \mathbb{X}_t, e_i \rangle = X^i_t$.
- The tensor components encode the **iterated integrals** of X

$$\langle \mathbb{X}_t, e_{ij} \rangle \text{``} = \text{''} \int_0^t \int_0^r dX^i_r dX^j_r$$

$$\text{and} \quad \langle \mathbb{X}_t, e_{ijk} \rangle \text{``} = \text{''} \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

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What if the integrals in equations like ($\dagger$) **don't** obey the usual laws of calculus?

Eg 1. Itô integrals.

Eg 2. Riemann-sum integrals for non-semimartingales (Burdzy, Swanson), Regularised integrals (Russo, Vallois).

This still fits into the framework of rough paths, but we need to add a few more **components** to $\mathbb{X}$.

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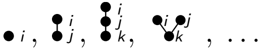
Non-geometric rough paths

Instead of tensors, the components of $\mathbb{X}$ are indexed by labelled **trees**



with the same labels used to index the basis of V.

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$$\bullet_i\,,\ \begin{smallmatrix}\bullet_i\\ |\\ \bullet_j\end{smallmatrix}\,,\ \begin{smallmatrix}\bullet_i\\ |\\ \bullet_j\\ |\\ \bullet_k\end{smallmatrix}\,,\ \begin{smallmatrix}\bullet_i & & \bullet_j\\ & \diagdown\diagup & \\ & \bullet_k & \end{smallmatrix}\,,\ \dots$$

with the same labels used to index the basis of V. And we have

$$\langle \mathbb{X}_t, \bullet_i \rangle = X^i_t \ , \qquad \langle \mathbb{X}_t, \begin{smallmatrix}\bullet_i\\ |\\ \bullet_j\end{smallmatrix} \rangle = \int_0^t \int_0^r dX^i_u dX^j_r$$

$$\langle \mathbb{X}_t, \begin{smallmatrix}\bullet_i\\ |\\ \bullet_j\\ |\\ \bullet_k\end{smallmatrix} \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r \ , \quad \langle \mathbb{X}_t, \begin{smallmatrix}\bullet_i & & \bullet_j\\ & \diagdown\diagup & \\ & \bullet_k & \end{smallmatrix} \rangle = \int_0^t X^i_r X^j_r dX^k_r$$

The object $\mathbb{X}$ is known as a **branched rough path** (Gubinelli).

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie. For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \dots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\
X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u]
\end{tikzcd}$$

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

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Consequences for rough DEs

Corollary (generalised Itô-Stratonovich correction formula)

Y is a solution to

$$dY_t = \sum_i f_i(Y_t) dX^i_t \quad \text{(driven by } \mathbb{X}\text{)}$$

if and only if Y is a solution to the rough DE

$$dY = \sum_i f_i(Y) \circ dX^i_t + \sum_\tau \bar{f}_\tau(Y_t) \circ d\bar{X}^\tau_t \quad \text{(driven by } \bar{\mathbb{X}}\text{)}\,,$$

where $\bar{\mathbb{X}}$ is the geometric rough path derived above.

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