Fast-slow systems with chaotic noise

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Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \, ,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x,y)\, \mu(dy) = 0$.

Our aim is to find a **reduced equation** for X.

Fast-slow systems

If we rescale to **large time scales** ($\sim \varepsilon^{-2}$) we have

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$
$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \ ,$$

We turn X_ε into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path X_ε as $\varepsilon \to 0$.

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon) v(Y_\varepsilon) + f(X_\varepsilon)$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(Y_\varepsilon(s))ds$ then

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

where the integral is of Riemann-Stieltjes type ($dW_\varepsilon = \frac{dW_\varepsilon}{ds} ds$).

Invariance principle for W_ε

We can write W_ε as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

We will call this class of Y **mildly chaotic**.

What about the SDE?

Since

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s))ds$$

But how should we interpret $\star dW$? Stratonovich? Itô? neither?

For **additive noise** $h(x) = I$
the answer is simple.

Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \,,$$

where U is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

Also works in the multiplicative noise case $(h(X)dU)$ but only when U is one dimensional.

The simple case (Melbourne, Stuart '11 + Gottwald, Melbourne'13)

If the flow is mildly chaotic ($W_\varepsilon \Rightarrow W$) then $X_\varepsilon \Rightarrow \bar{X}$ in the sup-norm topology, where

$$d\bar{X} = dW + f(\bar{X})ds \; .$$

In the multiplicative $1d$ noise case, the limit is Stratonovich

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds \; .$$

The strategy

The solution map takes "irregular path space" to "solution space"

$$\Phi : W_\varepsilon \mapsto X_\varepsilon$$

If this map were **continuous** then we could lift $W_\varepsilon \Rightarrow W$ to $X_\varepsilon \Rightarrow X$.

When the noise is both **multidimensional** and **multiplicative**, this strategy fails.

Ito, Stratonovich and family

SDEs are very **sensitive** wrt approximations of BM.

Suppose

$$dX = h(X)dW + f(X)dt$$

and define an approximation

$$dX_n = h(X_n)dW_n + f(X_n)dt$$

with some approximation W_n of W.

Taking $n \to \infty$, X_n might converge to something completely different to X. It all depends on the approximation W_n.

Eg. 1 If W_n is a step function approximation of W, then X_n converges to the **Ito** SDE

$$dX = h(X)dW + f(X)dt$$

Eg. 2 (Wong-Zakai) If W_n is a linear interpolation of W, then X_n converges to the **Stratonovich** SDE

$$dX = h(X) \circ dW + f(X)dt$$

Eg. 3 (McShane, Sussman) If W_n is a higher order interpolation of W, we can get limits which are **neither Ito nor Stratonovich**.

It is not enough to know that $W_n \to BM$.

We need more information.

Rough path theory (Lyons '97)

Provides a **unified** definition of a DE driven by a noisy path

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t h(X(s))ds$$

when the dU integral is not well defined.

In addition to U we must be given another path $\mathbb{U} : [0, T] \to \mathbb{R}^{d\times d}$ which is (formally) an iterated integral

$$\mathbb{U}^{ij}(t) \stackrel{def}{=} \int_0^t U^i(s)dU^j(s) \,.$$

These extra components tells us how to interpret the **method of integration**.

Rough path theory (Lyons '97)

Given a "rough path" $\mathbf{U} = (U, \mathbb{U})$ we can construct a solution

$$X(t) = X(0) + \int_0^t h(X(s))d\mathbf{U}(s) + \int_0^t h(X(s))ds$$

Eg. 1 If $U = W$ and $\mathbb{U} = \int W dW$ is the Ito iterated integral, then the constructed X is the solution to the Ito SDE.

Eg. 2 If $U = W$ and $\mathbb{U} = \int W \circ dW$ is the Stratonovich iterated integral, then the constructed X is the solution to the Stratonovich SDE.

Rough path theory (Lyons '97)

Most importantly (for us) the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an **extension** of the classical solution map and is **continuous** with respect to the "rough path topology".

Convergence of fast-slow systems

Returning to the slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

If we let

$$\mathbb{W}_\varepsilon^{ij}(t) = \int_0^t W_\varepsilon^i(r)dW_\varepsilon^j(r)$$

then $X_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon)$.

Due to the continuity of Φ, if $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$, then $X_\varepsilon \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

Theorem (K. & Melbourne '14)

If the fast dynamics are mildly chaotic, then $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s)dW^j(s) + \lambda^{ij} t$$

where the integral is Itô type and

$$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu\{v^i(Y(0))\, v^j(Y(s))\}\, ds\ .$$

$$\mathrm{Cov}^{ij}(W) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu\{v^i(Y(0))v^j(Y(s)) + v^j(Y(0))\, v^i(Y(s))\}\, ds$$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X_\varepsilon \Rightarrow \bar{X}$ *where*

$$d\bar{X} = h(\bar{X})dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt \; .$$

in Itô form, with λ^{ij} "$=$" $\int_0^\infty \mathbf{E}_\mu\{v^i(Y(0))\, v^j(Y(s))\}\, ds$

$$d\bar{X} = h(\bar{X}) \circ dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt$$

in Stratonovich form, with
λ^{ij} "$=$" $\int_0^\infty \mathbf{E}_\mu\{v^i(Y(0))\, v^j(Y(s)) - v^j(Y(0))\, v^i(Y(s))\}\, ds$.

Proof I : Find a martingale

The strategy is to decompose

$$W_\varepsilon(t) = M_\varepsilon(t) + A_\varepsilon(t)$$

where M_ε is a good martingale sequence (**Kurtz-Protter** 92)

$$\left(U_\varepsilon, M_\varepsilon, \int U_\varepsilon dM_\varepsilon\right) \Rightarrow \left(U, W, \int U dW\right)$$

where the integrals are of Itô type.

And $A_\varepsilon \to 0$ uniformly, but oscillates rapidly. Hence A_ε is like a corrector.

Proof II : Martingale approximation (Gordin 69)

Introduce a Poincaré section Λ with Poincaré map T and return times τ_j. Write

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \int_{\tau_j}^{\tau_{j+1}} v(Y(s))ds$$
$$= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \tilde{v}(T^j Y(0)) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} V_j \ .$$

We have a CLT sum for a stationary random sequence $\{V_j\}$ with natural filtration $\mathcal{F}_j = T^{-j}\mathcal{M}$ (where $\mathcal{M}$ is the σ-algebra for the $Y(0)$ probability space)

Proof II : Martingale approximation (Gordin 69)

Use a martingale approximation to show that $\varepsilon \sum_j^{N_\varepsilon - 1} V_j \Rightarrow W$.

Write $V_j = M_j + (Z_j - Z_{j+1})$ where $\mathbf{E}(M_j | \mathcal{F}_j) = 0$.

A good choice (if it converges) is the series

$$Z_j = \sum_{k=0}^{\infty} \mathbf{E}(V_{j+k} | \mathcal{F}_j) \ .$$

Convergence of this series is guaranteed by decay of correlations for the Poincaré map.

Proof II : Martingale approximation (Gordin 69)

The good martingale is $M_\varepsilon(t) = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j$ and the corrector is $A_\varepsilon(t) = \varepsilon(Z_0 - Z_{N_\varepsilon - 1})$. We then get

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) \Rightarrow W(t) + 0$$

by Martingale CLT and boundedness of Z.

We are sweeping a lot under the rug here since $\mathcal{F}_j \supseteq \mathcal{F}_{j+1}$. Need to reverse the martingales.

Proof III: Computing the iterated integral

To compute $\mathbb{W}_\varepsilon$ we decompose it

$$\int W_\varepsilon dW_\varepsilon = \int M_\varepsilon dM_\varepsilon + \int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dM_\varepsilon + \int A_\varepsilon dA_\varepsilon$$

Since M_ε is a good martingale sequence

$$\int M_\varepsilon dM_\varepsilon \Rightarrow \int W dW \quad \int A_\varepsilon dM_\varepsilon \Rightarrow 0 \ .$$

Even though $A_\varepsilon = O(\varepsilon)$, the iterated term $A_\varepsilon dA_\varepsilon$ does not vanish.
The last two terms are computed as ergodic averages

$$\int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dA_\varepsilon \to \lambda t \quad (a.s)$$

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. To appear in **Ann. Probab.** (2014).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2014).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. To appear in **Ann. App. Probab.** (2014).

All my slides are on my website (www.dtbkelly.com) **Thank you**!