

Fast-slow systems with chaotic noise

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Outline

Two problems :

- 1** - Fast-slow systems in continuous time
- 2** - Fast-slow systems in discrete time

Fast-slow systems in continuous time

Let $\dot{Y} = g(Y)$ be some chaotic flow with attractor Λ and invariant measure μ . We consider **fast-slow** systems of the form

$$\begin{aligned}\frac{dX^{(\varepsilon)}}{dt} &= \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)}) \\ \frac{dY^{(\varepsilon)}}{dt} &= \varepsilon^{-2} g(Y^{(\varepsilon)}),\end{aligned}$$

where $\varepsilon \ll 1$ and $h, f : \mathbb{R}^e \times \Lambda \rightarrow \mathbb{R}^e$ and $\int h(\cdot, y) \mu(dy) = 0$. Also assume that $Y(0) \sim \mu$.

The aim is to characterize the **distribution** of $X^{(\varepsilon)}$ as $\varepsilon \rightarrow 0$.

Fast-slow systems as SDEs

Consider the simplified **slow** equation

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1}h(X^{(\varepsilon)})v(Y^{(\varepsilon)}(t)) + f(X^{(\varepsilon)}(s))$$

where $h : \mathbb{R}^e \rightarrow \mathbb{R}^{e \times d}$ and $v : \Lambda \rightarrow \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t v(Y^{(\varepsilon)}(s))ds$ then

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

where the integral is of Riemann-Lebesgue type.

Invariance principle for $W^{(\varepsilon)}$

We can write $W^{(\varepsilon)}$ as

$$W^{(\varepsilon)}(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s)) ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s)) ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s)) ds$.

One can show that $W^{(\varepsilon)} \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

Continuity with respect to noise (Sussmann '78)

Suppose that

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds ,$$

where U is a smooth path.

If $d = 1$ or $h(x) = Id$ for all x , then $\Phi : U \rightarrow X$ is continuous in the sup-norm topology.

The simple case (Melbourne, Stuart '11)

If the flow is **chaotic** enough so that

$$W^{(\varepsilon)} \Rightarrow W ,$$

and **either** $d = 1$ **or** $h = \text{Id}$

then we have that $X^{(\varepsilon)} \Rightarrow X$ in the sup-norm topology, where

$$dX = h(X) \circ dW + f(X)ds ,$$

where the stochastic integral is of **Stratonovich** type.

This famously **falls apart** when the noise is both **multidimensional** and **multiplicative**. That is, when $d > 1$ **and** $h \neq Id$.

Continuity with respect to rough paths (Lyons '97)

As above, let

$$X(t) = \int_0^t h(X(s))dU(s) + \int_0^t F(X(s))ds ,$$

where U is a smooth path. Let $\mathbb{U} : [0, T] \rightarrow \mathbb{R}^{d \times d}$ be defined by

$$\mathbb{U}^{\alpha\beta}(t) \stackrel{\text{def}}{=} \int_0^t U^\alpha(s)dU^\beta(s) .$$

Then the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is **continuous** with respect to the " ρ_γ topology".

A general theorem for continuous fast-slow systems

Let $\mathbb{W}^{(\varepsilon),\alpha\beta}(t) = \int_0^t \mathbb{W}^{(\varepsilon),\alpha}(s) d\mathbb{W}^{(\varepsilon),\beta}(s)$.

Suppose that $(\mathbb{W}^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (\mathbb{W}, \mathbb{W})$ in the sup-norm topology where \mathbb{W} is Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t \mathbb{W}^\alpha(s) \circ d\mathbb{W}^\beta(s) + \lambda^{\alpha\beta} t$$

where $\lambda \in \mathbb{R}^{d \times d}$ and that $(\mathbb{W}^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)})$ satisfy the **tightness estimates**.

Then $X^{(\varepsilon)} \Rightarrow X$ in the sup norm topology, where

$$dX = h(X) \circ d\mathbb{W} + \left(f(X) + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h_j^k(X) \right) dt$$

Tightness estimates

To lift a sup-norm invariance principle to a ρ_γ invariance principle, we use the **Kolmogorov criterion**. Let

$$W^{(\varepsilon)}(s, t) = W^{(\varepsilon)}(t) - W^{(\varepsilon)}(s)$$
$$\mathbb{W}^{(\varepsilon), \alpha\beta}(s, t) = \int_s^t W^{(\varepsilon), \alpha}(s, r) dW^{(\varepsilon), \beta}(r)$$

The **tightness estimates** are of the form

$$(\mathbf{E}_\mu |W^{(\varepsilon)}(s, t)|^q)^{1/q} \lesssim |t-s|^\alpha \quad \text{and} \quad (\mathbf{E}_\mu |\mathbb{W}^{(\varepsilon)}(s, t)|^{q/2})^{2/q} \lesssim |t-s|^{2\alpha}$$

for q large enough and $\alpha > 1/3$.

We have the following result

Theorem (K, Melbourne '14)

If the *fast* dynamics are "*sufficiently chaotic*", then $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$ where W is a Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2} \lambda^{\alpha\beta} t$$

where

$$\lambda^{\beta\gamma} = \int_0^\infty \mathbf{E}_\mu(v^\beta v^\gamma(Y(s)) - v^\beta(Y(s)) v^\gamma) ds .$$

Homogenized equations

Corollary

Under the same assumptions as above, the *slow* dynamics $X^{(\varepsilon)} \Rightarrow X$ where

$$dX = h(X) \circ dW + \left(f(X) + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h_j^k(X) \right) dt .$$

Rmk. The only case where one gets Stratonovich is when the Auto-correlation is *symmetric*. For instance, if the flow is **reversible**.

Now let's try **discrete** time ...

Discrete time fast-slow systems

Suppose that $T : \Lambda \rightarrow \Lambda$ is a chaotic map with invariant measure μ . We consider the discrete fast-slow system

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2}h(X_j^{(n)}, T^j) + n^{-1}f(X_j^{(n)}, T^j)$$

Now define the path $X^{(n)}(t) = X_{\lfloor nt \rfloor}^{(n)}$.

The aim is to characterize the distribution of the path $X^{(n)}$ as $n \rightarrow \infty$.

Fast-slow systems as SDEs

Lets again simplify the slow equation to

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2} h(X_j^{(n)}) v(T^j) .$$

If we sum these up, we get

$$X^{(n)}(t) = X^{(n)}(0) + \sum_{j=0}^{\lfloor nt \rfloor - 1} h(X_j^{(n)}) \frac{v(T^j)}{n^{1/2}}$$

If we write $W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(T^j)$ then the path $X^{(n)}(t)$ satisfies

$$X^{(n)}(t) = X(0) + \int_0^t h(X^{(n)}(s-)) dW^{(n)}(s)$$

where the integral is defined in the “left-Riemann sum” sense.

Invariance principle

$$W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(T^j)$$

We still have that $W^{(n)} \Rightarrow W$ in the Skorokhod topology, where W is a multiple of Brownian motion.

But $W^{(n)}$ is a step function ... so RPT doesn't really work... even if it did, you'll never satisfy the tightness estimates.

A general theorem for discrete fast-slow systems (K 14')

Let

$$\mathbb{W}^{(n),\alpha\beta}(t) = n^{-1} \sum_{0 \leq i < j < \lfloor nt \rfloor} v^\alpha(T^i) v^\beta(T^j)$$

Suppose that $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the Skorokhod topology where W is Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \lambda^{\alpha\beta} t$$

where $\lambda \in \mathbb{R}^{d \times d}$ and that $(W^{(n)}, \mathbb{W}^{(n)})$ satisfy the **discrete tightness estimates**.

Then $X^{(n)} \Rightarrow X$ in the Skorokhod topology, where

$$dX(t) = h(X) \circ dW + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h_j^k(X) dt$$

Discrete tightness estimates

The discrete tightness estimates are a **coursier** version of the Kolmogorov criterion.

Let

$$W^{(n)}(s, t) = n^{-1/2} \sum_{\lfloor ns \rfloor \leq i < \lfloor nt \rfloor} v^\alpha(T^i) \quad \mathbb{W}^{(n)}(s, t) = n^{-1} \sum_{\lfloor ns \rfloor \leq i < j < \lfloor nt \rfloor} v^\alpha(T^i) v^\beta(T^j)$$

Then the **discrete tightness estimates** are of the form

$$\begin{aligned} (\mathbf{E}_\mu |W^{(n)}(\frac{j}{n}, \frac{k}{n})|^q)^{1/q} &\lesssim \left| \frac{j-k}{n} \right|^\alpha \quad \text{and} \\ (\mathbf{E}_\mu |\mathbb{W}^{(n)}(\frac{j}{n}, \frac{k}{n})|^{q/2})^{2/q} &\lesssim \left| \frac{j-k}{n} \right|^{2\alpha} \end{aligned}$$

for all $j, k = 0, \dots, n$, for q large enough and $\alpha > 1/3$.

We have the following result

Theorem (K, Melbourne '14)

If the *fast* dynamics are "*sufficiently chaotic*", then $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the Skorokhod topology, where W is a Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2} \kappa^{\alpha\beta} t$$

where

$$\kappa^{\alpha\beta} = \sum_{j=1}^{\infty} \mathbf{E}_\mu v^\alpha v^\beta(T^j)$$

Homogenized equations

Corollary

Under the same assumptions as above, the *slow* dynamics $X^{(n)} \Rightarrow X$ where

$$dX = h(X) \circ dW + \sum_{i,j,k} \frac{1}{2} \kappa^{jk} \partial^i h^j(X) h^{ik}(X) dt .$$

Idea of proof I

The idea is to find a rough path $\tilde{\mathbf{W}}^{(n)} = (\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)})$ such that

$$\left(\tilde{W}^{(n)}\left(\frac{j}{n}\right), \tilde{\mathbb{W}}^{(n)}\left(\frac{j}{n}\right) \right) = \left(W^{(n)}\left(\frac{j}{n}\right), \mathbb{W}^{(n)}\left(\frac{j}{n}\right) \right)$$

for all $j = 0, \dots, n$ and which is Lipschitz in between mesh points.

Then define

$$\tilde{X}^{(n)}(t) = X(0) + \int_0^t h(\tilde{X}^{(n)}(s)) d\tilde{\mathbf{W}}^{(n)}(s)$$

Idea of proof II

By construction, $\tilde{X}^{(n)}$ is a good approximation of $X^{(n)}$.

Proposition

We have that

$$\sup_{j=0\dots n} |X^{(n)}(j/n) - \tilde{X}^{(n)}(j/n)| \lesssim K_{n,\gamma} n^{1-3\gamma},$$

for any $\gamma \in (1/3, 1/2]$, where the constant $K_{n,\gamma}$ depends on n through the “discrete Hölder norms” of $(W^{(n)}, \mathbb{W}^{(n)})$.

As a consequence, if $\tilde{X}^{(n)} \Rightarrow X$ then $X^{(n)} \Rightarrow X$.

Idea of proof III

But since $\tilde{X}^{(n)}$ satisfies an RDE, we can apply the ideas from the first half of the talk.

But again by construction ...

- If $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the Skorokhod topology then $(\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the sup-norm topology.
- If $(W^{(n)}, \mathbb{W}^{(n)})$ satisfy the discrete tightness estimates, then $(\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)})$ satisfy the continuous tightness estimates.

Thus $\tilde{X}^{(n)} \Rightarrow X$.

