# Fast-slow systems with chaotic noise

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# Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$

$$\frac{dY}{dt} = g(Y) \, ,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space $\Lambda$ and ergodic invariant measure $\mu$. (**eg**. 3$d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y) \, \mu(dy) = 0$.

Our aim is to find a **reduced equation** $\frac{d\bar{X}}{dt} = F(\bar{X})$ with $\bar{X} \approx X$.

# Fast-slow systems

If we rescale to **large time scales** we have

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$

$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \,,$$

We turn $X_\varepsilon$ into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path $X_\varepsilon$ as $\varepsilon \to 0$.

Why is model reduction important?

**1** - The reduced model is **lower dimensional** and **less stiff** than the original fast-slow system.

**2** - Helps the user make **informed guess** when the model is unknown.

# Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon) v(Y_\varepsilon) + f(X_\varepsilon)$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(Y_\varepsilon(s))ds$ then

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

where the integral is of Riemann-Stieltjes type ($dW_\varepsilon = \frac{dW_\varepsilon}{ds}ds$).

# Invariance principle for $W_\varepsilon$

We can write $W_\varepsilon$ as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on $Y$ lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic $Y$, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where $W$ is a multiple of Brownian motion.

We will call this class of $Y$ **mildly chaotic**.

# What about the SDE?

Since

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s))ds$$

But how should we interpret $\star dW$ ? Stratonovich? Itô? neither?

For **additive noise** $h(x) = I$
the answer is simple.

# Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \, ,$$

where $U$ is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

# The simple case (Melbourne, Stuart '11)

If the flow is mildly chaotic ($W_\varepsilon \Rightarrow W$) then $X_\varepsilon \Rightarrow \bar{X}$ in the sup-norm topology, where

$$d\bar{X} = dW + f(\bar{X})ds \ .$$

Same idea works for general $h$ provided $n = d = 1$. The limit is Stratonovich

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds \ .$$

# The strategy

The solution map takes "noisy path space" to "solution space"

$$\Phi : W_\varepsilon \mapsto X_\varepsilon$$

If this map were **continuous** then we could lift $W_\varepsilon \Rightarrow W$ to $X_\varepsilon \Rightarrow X$.

When the noise is both **multidimensional** and **multiplicative**, this strategy fails.

# Ito, Stratonovich and family

SDEs are very **sensitive** wrt approximations of BM.

Suppose
$$dX = h(X)dW + f(X)dt$$
and define an approximation
$$dX_n = h(X_n)dW_n + f(X_n)dt$$
with some approximation $W_n$ of $W$.

Taking $n \to \infty$, $X_n$ might converge to something completely different to $X$. It all depends on the approximation $W_n$.

**Eg. 1** If $W_n$ is a step function approximation of $W$, then $X_n$ converges to the **Ito** SDE

$$dX = h(X)dW + f(X)dt$$

**Eg. 2** (Wong-Zakai) If $W_n$ is a linear interpolation of $W$, then $X_n$ converges to the **Stratonovich** SDE

$$dX = h(X) \circ dW + f(X)dt$$

**Eg. 3** (Sussman) If $W_n$ is a higher order spline interpolation of $W$, we can get limits which are **neither Ito nor Stratonovich**.

It is not enough to know that
$W_n \to BM$.

We need more information.

# Rough path theory (Lyons '97)

Provides a **unified** definition of a DE driven by a noisy path

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t h(X(s))ds$$

when the $dU$ integral is not well defined.

In addition to $U$ we must be given another path $\mathbb{U} : [0, T] \to \mathbb{R}^{d \times d}$ which is (formally) an iterated integral

$$\mathbb{U}^{ij}(t) \overset{def}{=} \int_0^t U^i(s)dU^j(s) .$$

These extra components tells us how to interpret the **method of integration**.

# Rough path theory (Lyons '97)

Given a "rough path" $\mathbf{U} = (U, \mathbb{U})$ we can construct a solution

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{U}(s) + \int_0^t h(X(s)) ds$$

**Eg. 1** If $U = W$ and $\mathbb{U} = \int W dW$ is the Ito iterated integral, then the constructed $X$ is the solution to the Ito SDE.

**Eg. 2** If $U = W$ and $\mathbb{U} = \int W \circ dW$ is the Stratonovich iterated integral, then the constructed $X$ is the solution to the Stratonovich SDE.

# Rough path theory (Lyons '97)

Most importantly (for us) the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an extension of the classical solution map and is **continuous** with respect to the "rough path topology".

# Convergence of fast-slow systems

Returning to the slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

If we let

$$\mathbb{W}_\varepsilon^{ij}(t) = \int_0^t W_\varepsilon^i(r)dW_\varepsilon^j(r)$$

then $X_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon)$.

Due to the continuity of $\Phi$, if $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$, then $X_\varepsilon \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

## Theorem (K. & Melbourne '14)

*If the fast dynamics are mildly chaotic, then $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$
where $W$ is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s)\, dW^j(s) + \lambda^{ij} t$$

*where the integral is Itô type and*

$$\lambda^{ij} \text{ "} = \text{"} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds \ .$$

$$\mathrm{Cov}^{ij}(W) \text{ "} = \text{"} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0)) v^j(Y(s)) + v^j(Y(0))\, v^i(Y(s))) \}\, ds$$

# Homogenized equations

## Corollary

*Under the same assumptions as above, the slow dynamics $X_\varepsilon \Rightarrow \bar{X}$ where*

$$d\bar{X} = h(\bar{X})dW + \left( f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt \ .$$

*in Itô form, with $\lambda^{ij} \text{ " = " } \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0)) \, v^j(Y(s)) \} \, ds$*

$$d\bar{X} = h(\bar{X}) \circ dW + \left( f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt$$

*in Stratonovich form, with*
*$\lambda^{ij} \text{ " = " } \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0)) \, v^j(Y(s)) - v^j(Y(0)) \, v^i(Y(s)) \} \, ds \ .$*

What about the original (much more complicated) fast-slow system?

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$

$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \ .$$

# General fast-slow systems II

The slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) ds + \int_0^t f(X_\varepsilon, Y_\varepsilon) ds$$

can be written in the **product form**

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t H(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t H(X_\varepsilon(s)) dV_\varepsilon(s)$$

$H$ is the **evaluation map** (or Dirac distribution) $H(x)\varphi = \varphi(x)$ for $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ suitably smooth. And $W_\varepsilon, V_\varepsilon$ are the **function valued paths**

$$W_\varepsilon(t) = \varepsilon^{-1} \int_0^t h(\cdot, Y_\varepsilon(s)) ds \quad V_\varepsilon(t) = \int_0^t f(\cdot, Y_\varepsilon(s)) ds$$

# General fast-slow systems III

**Theorem (K. & Melbourne '14)**

*If the fast dynamics are mildly chaotic then $X_\varepsilon \Rightarrow \bar{X}$ where*

$$d\bar{X} = \sigma(\bar{X})dB + \tilde{a}(\bar{X})dt \,,$$

*where $B$ is a standard BM on $\mathbb{R}^d$ and*

$$\tilde{a}(x) = \int f(x,y)d\mu(y) + \sum_{k=1}^{d} \mathfrak{B}(h^k(x,\cdot), \partial_k h(x,\cdot))$$

$$\sigma\sigma^T(x) = \mathfrak{B}(h^i(x,\cdot), h^j(x,\cdot)) + \mathfrak{B}(h^j(x,\cdot), h^i(x,\cdot))$$

*and $\mathfrak{B}$ is the "integrated autocorrelation" of the fast dynamics*

$$\mathfrak{B}(v,w)\text{"}=\text{"} \int_0^\infty \mathbf{E}_\mu v(Y(0))w(Y(s))ds$$

# The real world has feedback

It is more realistic to look fast-slow systems of the form

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1}h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$

$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2}g(Y_\varepsilon) + \varepsilon^{\beta-2}g_0(X_\varepsilon, Y_\varepsilon) \,,$$

for some $\beta \geq 1$. Since the coupling term is of lower order, this is called **weak feedback**.

**Back of the envelope**: For $\beta > 1$, the reduced model is exactly the same as the the zero feedback case.

For $\beta = 1$, an additional correction term appears, which involves the weak feedback term $g_0$.

# The real world is infinite dimensional

Many fast-slow models are **PDEs**.

Suppose that $Y_\varepsilon = (Y_\varepsilon^1, Y_\varepsilon^2, \dots)$ is an infinite vector of fast, chaotic variables (possibly coupled). Can we identify a reduced model for $X_\varepsilon = X_\varepsilon(t, x)$ where

$$\partial_t X_\varepsilon = \Delta X_\varepsilon + \varepsilon^{-1} H(X_\varepsilon, Y_\varepsilon) + F(X_\varepsilon, Y_\varepsilon)$$

This is a delicate question, since many natural approximations of noise yield **infinites** in the limiting SPDE.

This is a problem for Hairer's theory of **regularity structures**.

# References

**1** - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. To appear in **Ann. Probab.** (2014).

**2** - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2014).

**3** - D. Kelly. *Rough path recursions and diffusion approximations*. To appear in **Ann. App. Probab.** (2014).

All my slides are on my website (www.dtbkelly.com) **Thank you**!