Fast-slow systems with chaotic noise

David Kelly Ian Melbourne

Courant Institute
New York University
New York NY
www.dtbkelly.com

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Weak invariance principles

Suppose we have a deterministic flow $\phi_t : \Lambda \to \Lambda$ with ergodic invariant measure ν and an observable $v : \Lambda \to \mathbb{R}^n$. Define

$$W_\varepsilon(t)(\omega) = \varepsilon \int_0^{\varepsilon^{-2} t} v \circ \phi_s(\omega) ds$$

where $\omega \sim \nu$. A weak invariance principle (WIP) describes the convergence result

$$W_n \to_w W$$

in C^0, where W is a multiple of Brownian motion in $\mathbb{R}^n$.

WIPs are known for large classes of uniformly and non-uniformly hyperbolic flows.

Can we use WIPs for homogenization of deterministic fast-slow systems?

Fast-slow with additive noise

Consider the fast-slow system

$$\dot{x}_\varepsilon = \varepsilon^{-1} v(y_\varepsilon) + f(x_\varepsilon)$$
$$\dot{y}_\varepsilon = \varepsilon^{-2} g(y_\varepsilon)$$

where $y_\varepsilon(t) = y(\varepsilon^{-2} t)$, $y(0) \sim \nu$ and $\dot{y} = g(y)$ generates a flow ϕ_t that has a WIP.

Thm (Melbourne, Stuart 11): $x_\varepsilon \to_w X$ (in the sup-norm topology) where X satisfies the SDE

$$dX = dW + f(X)dt$$

Can be extended to multiplicative noise

$$\dot{x}_\varepsilon = \varepsilon^{-1} h(x_\varepsilon) v(y_\varepsilon) + f(x_\varepsilon)$$

provided that $h = \nabla V$, and extended to discrete-time formulations (**Gottwald, Melbourne 14**).

Proof

Simple fact: Suppose that W is a uniformly continuous path and that X solves the integral ODE

$$X(t) = X(0) + W(t) + \int_0^t f(X(s))ds$$

Then the map $\Phi : W \mapsto X$, $(C^0 \to C^0)$ is continuous.

The slow variables satisfy the integral equation

$$\begin{aligned} x_\varepsilon(t) &= x(0) + \varepsilon^{-1} \int_0^t v(y_\varepsilon(s))ds + \int_0^t f(x_\varepsilon(s))ds \\ &= x(0) + W_\varepsilon(t) + \int_0^t f(x_\varepsilon(s))ds \end{aligned}$$

By **cts mapping thm**, if $W_\varepsilon \to_w W$ then $x_\varepsilon = \Phi(W_\varepsilon) \to_w \Phi(W) = X$.

More complicated fast-slow systems?

1 Continuous time with non-trivial products

$$\dot{x}_\varepsilon = \varepsilon^{-1} h(x_\varepsilon) v(y_\varepsilon) + f(x_\varepsilon)$$

2 Continuous time with non-products

$$\dot{x}_\varepsilon = \varepsilon^{-1} h(x_\varepsilon, y_\varepsilon) + f(x_\varepsilon, y_\varepsilon)$$

3 Discrete-time with non-trivial products

$$x_{j+1}^{(n)} = x_j^{(n)} + n^{-1/2} h(x_j^{(n)}) v(y_j) + n^{-1} f(x_j^{(n)})$$
$$y_{j+1} = T y_j$$

and let $x_n(t) := x_{\lfloor nt \rfloor}^{(n)}$

Non-trivial product case

$$\dot{x}_\varepsilon = \varepsilon^{-1} h(x_\varepsilon) v(y_\varepsilon)$$

$(v : \Lambda \to \mathbb{R}^n\ ,\ h : \mathbb{R}^d \to \mathbb{R}^{d\times n}$, take $f = 0)$

Since $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(y_\varepsilon(s))ds$, we can write the slow variables in the integral form

$$x_\varepsilon(t) = x(0) + \int_0^t h(x_\varepsilon(s))dW_\varepsilon(s)$$

where dW_ε denotes **Lebesgue-Stieltjes integration** ($dW_\varepsilon = \dot{W}_\varepsilon ds$).

Can we apply the same continuity trick? The map Φ is defined on BV paths, but W_ε converges in a much weaker topology, so we must extend Φ to a bigger space of paths.

Attempted construction of Φ

First define the map $\Phi : W \to X$ where W is smooth and

$$X(t) = X(0) + \int_0^t h(X(s))dW(s) \,,$$

with the integral interpreted in Lebesgue-Stieltjes sense.

Extend the map Φ to the closure of the space of smooth paths under the C^γ Hölder topology, for some $\gamma \in (1/3, 1/2]$. ie. $\Phi(W) = \lim_{n\to\infty} \Phi(W_n)$ where W_n is a smooth approx of $W \in C^\gamma$.

This map is **not well-posed**, the limiting X depends on the choice of sequence W_n.

Counter-example to well-posedness

Let $X = (X^1, X^2, X^3)$ be defined by

$$X^1(t) = \int_0^t dZ^1 \qquad X^2(t) = \int_0^t dZ^2$$

$$X^3(t) = \int_0^t X^2 dZ^1 - \int_0^t X^1 dZ^2$$

where $Z = (0,0,0)$. Since Z is smooth, we should clearly have $X = (0,0,0)$.

But we could equally take the approximation $Z_n = (n^{-1}\cos(n^2 t), n^{-1}\sin(n^2 t))$ for $0 \leq t \leq 2\pi$. Clearly $Z_n \to 0$ in sup-norm topology (actually in C^γ for all $\gamma < 1/2$). But

$$X^3(t) = \int_0^t Z_n^2 dZ_n^1 - \int_0^t Z_n^1 dZ_n^2 = 2\times \text{ area inside the curve } (Z_n^1, Z_n^2)$$

Thus $X_n(2\pi) \to (0,0,2\pi)$.

Rough path theory (Lyons 98)

Consider the space of pairs $(W, \mathbb{W}) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^{n\times n}$ where W is a smooth path and $\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) dW^\beta(s)$ is the **iterated integral** of W.

The space of γ-**rough paths** $\mathscr{C}^\gamma$ is the closure of the above space under the γ-Hölder metric

$$\rho_\gamma(W, \mathbb{W}; \tilde{W}, \tilde{\mathbb{W}}) = \sup_{s,t} \frac{|W(s,t) - \tilde{W}(s,t)|}{|s-t|^\gamma} + \sup_{s,t} \frac{|\mathbb{W}(s,t) - \tilde{\mathbb{W}}(s,t)|}{|s-t|^{2\gamma}}$$

where $W(s,t) = W(t) - W(s)$ and $\mathbb{W}(s,t) = \int_s^t W(s,r) dW(r)$.

Rough path theory (Lyons 98)

Define Φ on the space of smooth pairs by $\Phi(W, \mathbb{W}) = X$ where

$$X(t) = X(0) + \int_0^t h(X(s))dW(s)$$

Then extend Φ to rough paths $\mathscr{C}^\gamma$ as $\Phi(W, \mathbb{W}) = \lim_{n\to\infty} \Phi(W_n, \mathbb{W}_n)$, where $(W_n, \mathbb{W}_n)$ is a smooth approx of $(W, \mathbb{W})$ in $\mathscr{C}^\gamma$.

Thm: The map $\Phi : \mathscr{C}^\gamma \to C^0$ is well-defined and continuous.

Rough paths for fast-slow systems

Corollary: Let x_ε solve

$$x_\varepsilon(t) = x(0) + \int_0^t h(x_\varepsilon(s)) dW_\varepsilon(s)$$

where W_ε is a smooth and let $\mathbb{W}_\varepsilon = \int W_\varepsilon dW_\varepsilon$. Suppose that

(i) $(W_\varepsilon, \mathbb{W}_\varepsilon) \to_w (W, \mathbb{W})$ in C^0, where W is a BM and $\mathbb{W}(t) = \int_0^t W dW + \lambda t$, where the integral is of Itô type.

(ii) $(\mathbf{E}|W_\varepsilon(s,t)|^q)^{1/q} \lesssim |t-s|^{1/2}$ and $\left(\mathbf{E}|\mathbb{W}_\varepsilon(s,t)|^{2q}\right)^{1/(2q)} \lesssim |t-s|$ uniformly in s, t, ε with $q > 3$.

Then $x_\varepsilon \to_w X$ in C^0 where X satisfies the SDE

$$dX = h(X) dW + \sum_{\alpha,\beta,\kappa} \lambda^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X) dt \, .$$

Proof of Corollary

The iterated WIP + the Kolmogorov estimates yield $(W_\varepsilon, \mathbb{W}_\varepsilon) \to_w (W, \mathbb{W})$ in the rough path topology.

Since Φ is continuous wrt this topology, we have

$$x_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon) \to_w \Phi(W, \mathbb{W}) \; .$$

Standard result from rough path theory: if W BM and $\mathbb{W}(t) = \int_0^t W dW + \lambda t$ then $X = \Phi(W, \mathbb{W})$ satisfies the SDE

$$dX = h(X)dW + \sum_{\alpha,\beta,\kappa} \lambda^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X) dt \; .$$

□

Iterated WIP

Suppose that $\phi_t : \Omega \to \Omega$ is a suspension flow, with Poincaré map T that is modeled by a mixing Young tower whose return times have sufficiently decaying tails.

Thm (Kelly, Melbourne 16'): $(W_\varepsilon, \mathbb{W}_\varepsilon) \to_w (W, \mathbb{W})$ in C^0, where W is a BM with covariance Σ, $\mathbb{W} = \int W dW + \lambda t$ and

$$\Sigma^{\alpha\beta} = \lim_{\varepsilon \to 0} \mathbf{E}_\nu W_\varepsilon^\alpha(1) W_\varepsilon^\beta(1) \text{ “}=\text{” } \int_0^\infty \mathbf{E}_\nu (v^\alpha v^\beta \circ \phi_s + v^\beta v^\alpha \circ \phi_s) ds$$

and

$$\lambda^{\alpha\beta} = \lim_{\varepsilon \to 0} \mathbf{E}_\nu \mathbb{W}_\varepsilon^{\alpha\beta}(1) \text{ “}=\text{” } \int_0^\infty \mathbf{E}_\nu v^\alpha v^\beta \circ \phi_s ds$$

(“ = ” holds if the integrals exist)

Corollary (Kelly, Melbourne 16') Under the above hypotheses $x_\varepsilon \to_w X$ in C^0, where

$$dX = h(X)dW + \sum_{\alpha,\beta,\kappa} \lambda^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X)dt$$

and λ is as above. Or in Stratonovich form

$$dX = h(X) \circ dW + \sum_{\alpha,\beta,\kappa} \theta^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X)dt$$

where

$$\theta^{\alpha\beta} = 2\lambda^{\alpha\beta} - \Sigma^{\alpha\beta} \text{ “} = \text{”} \int_0^\infty \mathbf{E}_\nu (v^\alpha v^\beta \circ \phi_s - v^\beta v^\alpha) \circ \phi_s ds$$

Proof I : Find a martingale

The strategy is to decompose

$$W_\varepsilon(t) = M_\varepsilon(t) + A_\varepsilon(t)$$

where M_ε is a good martingale sequence (**Kurtz-Protter** 92):
If $(U_\varepsilon, M_\varepsilon) \to_w (U, W)$ then

$$\left(U_\varepsilon, M_\varepsilon, \int U_\varepsilon dM_\varepsilon\right) \Rightarrow \left(U, W, \int U dW\right)$$

where the integrals are of Itô type.

And $A_\varepsilon \to 0$ uniformly, but oscillates rapidly. Hence A_ε is like a corrector.

Proof II : Martingale approximation

Let τ_j be the j-th return time for the Poincaré map T, then

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \int_{\tau_j}^{\tau_{j+1}} v \circ \phi_s ds + A_{\varepsilon,1}(t)$$
$$= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \tilde{v} \circ T^j + A_{\varepsilon,1}(t)$$

where $A_{\varepsilon,1} = O(\varepsilon)$ in probability.

Apply martingale-coboundary decomposition $\tilde{v} = m + \chi \circ T - \chi$

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} m \circ T^j + \varepsilon(\chi \circ T^{N(\varepsilon^{-2}t)} - \chi) + A_{\varepsilon,1}(t)$$
$$= M_\varepsilon(t) + A_\varepsilon(t)$$

Proof III: Compute the integrals

Decompose the iterated integral

$$\int W_\varepsilon dW_\varepsilon = \int W_\varepsilon dM_\varepsilon + \int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dA_\varepsilon$$

As a good martingale seqeuence dM_ε converges to the Itô integral $\int W dW$.

By ergodic thm, we can compute

$$\int A_\varepsilon dM_\varepsilon \to 0 \qquad \int A_\varepsilon dA_\varepsilon \to \lambda t$$

Even though $A_\varepsilon \to 0$ uniformly, its iterated integral does not. ■

Non-product case

$$\dot{x}_\varepsilon = \varepsilon^{-1} h(x_\varepsilon, y_\varepsilon) + f(x_\varepsilon, y_\varepsilon)$$

$$h, f : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$$

Non-product = infinite dimensional product

Let $\mathcal{B}$ be some Banach space of functions $\varphi : \mathbb{R}^d \to \mathbb{R}^d$.
Define the paths $W_\varepsilon, V_\varepsilon : [0, T] \to \mathcal{B}$

$$W_\varepsilon(t) = \varepsilon^{-1} \int_0^t h(\cdot, y_\varepsilon(s)) ds \qquad V_\varepsilon(t) = \int_0^t f(\cdot, y_\varepsilon(s)) ds$$

and define the linear functional $H : \mathcal{B} \to \mathbb{R}^d$ by $H(x)\varphi = \varphi(x)$ (ie. $H(x)$ corresponds to integration against δ_x). Then we have

$$x_\varepsilon(t) = x(0) + \int_0^t H(x_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t H(x_\varepsilon(s)) dV_\varepsilon(s)$$

and rough path theory works for Banach space valued paths!

Thm (Kelly, Melbourne 15). Under the same assumptions on ϕ_t, $x_\varepsilon \to_w X$ where

$$dX = \sigma(X)dB + b(X)dt$$

where B is a standard BM on $\mathbb{R}^d$ and

$$b(x) = \int f(x,y)d\nu(y) + \sum_{k=1}^{d} \mathfrak{B}(h^k(x,\cdot), \partial_k h(x,\cdot))$$
$$\sigma\sigma^T(x) = \mathfrak{B}(h^i(x,\cdot), h^j(x,\cdot)) + \mathfrak{B}(h^j(x,\cdot), h^i(x,\cdot))$$

and $\mathfrak{B}$ is the "integrated autocorrelation" of the fast dynamics

$$\mathfrak{B}(v,w) = \mathbf{E}_\nu W_{\varepsilon,v}(1) W_{\varepsilon,w}(1) \text{ “}=\text{” } \int_0^\infty \mathbf{E}_\nu v \, w \circ \phi_s ds$$

and $W_{\varepsilon,v} = \varepsilon \int_0^{\varepsilon^{-2}t} v \circ \phi_s ds$, $v : \Omega \to \mathbb{R}$

Discrete time case

$$x_{j+1}^{(n)} = x_j^{(n)} + n^{-1/2} h(x_j^{(n)}) v(y_j)$$
$$y_{j+1} = T y_j$$

where $T : \Lambda \to \Lambda$, ergodic invariant measure μ and let
$$x_n(t) := x_{\lfloor nt \rfloor}^{(n)}$$

Notice that x_n satisfies the 'summation equation'

$$x_n(t) = x(0) + \sum_{j=0}^{\lfloor nt \rfloor - 1} h(x_n(j/n)) n^{-1/2} v \circ T^j$$

which we can write as a Left-Riemann sum

$$x_n(t) = x(0) + \int_0^t h(x_n(s-)) dW_n(s)$$

where $W_n(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v \circ T^j$.

There are large classes of maps for which $W_n \to_w W$, where W is a multiple of BM. Can we use rough path theory?

Define the iterated sum

$$\mathbb{W}_n^{\alpha\beta}(t) = n^{-1} \sum_{j=0}^{\lfloor nt \rfloor -1} \sum_{i=0}^{j-1} v^\alpha \circ T^i v^\beta \circ T^j$$
$$= \int_0^t W^\alpha(s-) dW^\beta(s)$$

Can we use an iterated WIP for the discrete-time rough path $(W_n, \mathbb{W}_n)$ to obtain homogenization for x_n?

Thm (Kelly 15): Let x_n solve

$$x_n(t) = x(0) + \int_0^t h(x_n(s-))dW_n(s)$$

where $(W_n, \mathbb{W}_n)$ are the cadlag step functions as above. Suppose that

(i) $(W_n, \mathbb{W}_n) \to_w (W, \mathbb{W})$ in the Skorokhod topology, where W is a BM and $\mathbb{W}(t) = \int_0^t W dW + \lambda t$, where the integral is of Itô type.

(ii) $(\mathbf{E}|W_n(s,t)|^q)^{1/q} \lesssim |t-s|^{1/2}$ and $\left(\mathbf{E}|\mathbb{W}_n(s,t)|^{2q}\right)^{1/(2q)} \lesssim |t-s|$ uniformly in $s,t \in \{j/n : 0 \le j \le n-1\}$ and $n \ge 1$ with $q > 3$.

Then $x_\varepsilon \to_w X$ in the Skorokhod topology, where X satisfies the SDE

$$dX = h(X)dW + \sum_{\alpha,\beta,\kappa} \lambda^{\alpha\beta}\partial_\kappa h^\alpha(X)h^{\beta\kappa}(X)dt\,.$$

Proof ideas I

Let $\tilde{W}_n$ be a smooth interpolation of the step-function W_n (for instance, linear interpolation) and let $\tilde{\mathbb{W}}_n = \int \tilde{W}_n d\tilde{W}_n$. If we define

$$\tilde{x}_n(t) = x(0) + \int_0^t h(\tilde{x}_n(s)) d\tilde{W}_n(s) + \int_0^t \partial_\kappa h^\alpha(\tilde{x}_n(s)) h^{\beta\kappa} d\tilde{Z}_n^{\alpha\beta}(s) .$$

where $\tilde{Z}_n = \mathbb{W}_n - \tilde{\mathbb{W}}_n$, then one can show $\|\tilde{x}_n - x_n\|_\infty = o(1)$ in probability. Similar to **backward error analysis** for numerical methods.

Proof ideas II

The assumptions on $(W_n, \mathbb{W}_n)$ guarantee that

$$(\tilde{W}_n, \tilde{\mathbb{W}}_n; \tilde{Z}_n) \to_w (W, \tilde{\mathbb{W}}; \tilde{\lambda} t)$$

in the rough path topology, where $\tilde{\mathbb{W}} = \mathbb{W} - \tilde{\lambda} t = \int W dW + (\lambda - \tilde{\lambda}) t$.

We obtain $\tilde{x}_n = \Phi(\tilde{W}_n, \tilde{\mathbb{W}}_n; Z_n) \to_w \Phi(W, \mathbb{W} - \tilde{\lambda} t; \tilde{\lambda} t) = X$ and hence

$$\begin{aligned} dX = h(X) dW &+ \sum_{\alpha,\beta,\kappa} (\lambda - \tilde{\lambda})^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X) dt \\ &+ \sum_{\alpha,\beta,\kappa} \tilde{\lambda}^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X) dt \end{aligned}$$

■

Suppose T is modeled by a mixing Young tower whose return times have sufficiently decaying tails.

Thm (Kelly, Melbourne 16') $x_n \to_w X$ in C^0, where

$$dX = h(X)dW + \sum_{\alpha,\beta,\kappa} \lambda^{\alpha\beta} \partial_\kappa h^\alpha(X) h^{\beta\kappa}(X) dt$$

and

$$\lambda^{\alpha\beta} = \sum_{n \geq 1} \mathbf{E}_\nu v^\alpha v^\beta \circ T^n$$

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. **Ann. Probab.** (2016).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2015).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. **Ann. App. Probab.** (2015).

All my slides and **lecture notes** from last week's course are/will be on my website (www.dtbkelly.com) **Thank you**!