Fast-slow systems with chaotic noise

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November 20, 2014

Applied math seminar, CIMS.

Fast-slow systems

Let $\dot{Y} = g(Y)$ be some weakly chaotic ODE with state space Λ and ergodic invariant measure μ. We consider fast-slow systems of the form

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$
$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) \; ,$$

where $\varepsilon \ll 1$ and $h, f : \mathbb{R}^e \times \Lambda \to \mathbb{R}^e$ and $\int h(\cdot, y)\, \mu(dy) = 0$. Also assume that $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of $X^{(\varepsilon)}$ as $\varepsilon \to 0$.

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}) v(Y^{(\varepsilon)}) + f(X^{(\varepsilon)})$$

where $h : \mathbb{R}^e \to \mathbb{R}^{e \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t v(Y^{(\varepsilon)}(s))ds$ then

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

where the integral is of Riemann-Lebesgue type.

Invariance principle for $W^{(\varepsilon)}$

We can write $W^{(\varepsilon)}$ as

$$W^{(\varepsilon)}(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

One can show that $W^{(\varepsilon)} \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

What about the SDE?

Since

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

This suggest a limiting SDE

$$X(t) = X(0) + \int_0^t h(X(s)) \star dW(s) + \int_0^t f(X(s))ds$$

But how should we interpret $\star dW$?

Continuity with respect to noise (Sussmann '78)

Suppose that

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds \,,$$

where U is a uniformly continuous path.

If $d = 1$ **or** $h(x) = Id$ for all x, then $\Phi : U \to X$ is continuous in the sup-norm topology.

The simple case (Melbourne, Stuart '11)

If the flow is chaotic enough so that

$$W^{(\varepsilon)} \Rightarrow W \;,$$

and **either** $d = 1$ **or** $h = \mathrm{Id}$

then we have that $X^{(\varepsilon)} \Rightarrow X$ in the sup-norm topology, where

$$dX = h(X) \circ dW + f(X) ds \;,$$

where the stochastic integral is of Stratonovich type.

Continuity of the solution map

The solution map takes "noisy path space" to "solution space"

$$\Phi : W^{(\varepsilon)} \mapsto X^{(\varepsilon)}$$

If this map were **continuous** then we could lift $W^{(\varepsilon)} \Rightarrow W$ to $X^{(\varepsilon)} \Rightarrow X$.

Continuity of the solution map

We want to define a map $\Phi : U \to X$ where U is a noisy path and

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds$$

This is problematic for two reasons.

1 - The solution map Φ is only defined for *differentiable* noise. But $W^{(\varepsilon)} \Rightarrow W$ and Brownian motion is *not differentiable*.

2 - Any attempt to define an extension of Φ to Brownian-like objects will fail to be continuous. ie. We can find a sequence $W_n \Rightarrow W$ but $\Phi(W_n) \not\Rightarrow \Phi(W)$.

The lesson is, we must use extra information about the noise to construct a continuous extension.

Rough path theory (Lyons '97)

Suppose we are given a path $\mathbb{U} : [0, T] \to \mathbb{R}^{d\times d}$ which is (formally) an iterated integral

$$\mathbb{U}^{ij}(t) \stackrel{def}{=} \int_0^t U^i(s) dU^j(s) \; .$$

Given a "rough path" $\mathbf{U} = (U, \mathbb{U})$ we can construct a solution

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{U}(s) + \int_0^t h(X(s)) ds$$

The map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an extension of the classical solution map and is **continuous** with respect to the "rough path topology".

Convergence of fast-slow systems

If we let

$$\mathbb{W}^{ij,(\varepsilon)}(t) = \int_0^t W^{i,(\varepsilon)}(r) dW^{j,(\varepsilon)}(r)$$

then $X^{(\varepsilon)} = \Phi(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)})$.

Due to the continuity of Φ, if $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$, then $X^{(\varepsilon)} \Rightarrow X$, where

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{W}(s) + \int_0^t h(X(s)) ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

We have the following result

Theorem (K. & Melbourne)

If the fast dynamics are "sufficiently chaotic", then $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s)dW^j(s) + \lambda^{ij} t$$

where the integral is Ito type and

$$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu(v^i\, v^j(Y(s))\, ds\ .$$

$$\mathrm{Cov}^{ij}(W) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu(v^i\, v^j(Y(s)) + v^j\, v^i(Y(s)))\, ds\ .$$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X^{(\varepsilon)} \Rightarrow X$ *where*

$$dX = h(X)dW + \left(f(X) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(X) h^{kj}(X) \right) dt \, .$$

General fast-slow systems I

The original fast-slow system was

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$
$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) \ .$$

How can we write this as an "approximate SDE" when h is not a product?

General fast-slow systems II

Let H be the evaluation map (or Dirac distribution) $H(x)\varphi = \varphi(x)$ for $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ suitably smooth.

Let us define the **infinite dimensional paths**

$$W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t h(\cdot, Y^{(\varepsilon)}(s))ds \quad V^{(\varepsilon)}(t) = \int_0^t f(\cdot, Y^{(\varepsilon)}(s))ds$$

then

$$H(X^{(\varepsilon)})dW^{(\varepsilon)} = H(X^{(\varepsilon)})\varepsilon^{-1}h(\cdot, Y^{(\varepsilon)})dt = \varepsilon^{-1}h(X^{(\varepsilon)}, Y^{(\varepsilon)})dt$$

and similarly for $H(X^{(\varepsilon)})dV^{(\varepsilon)}$.It follows that we can write

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t H(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t H(X^{(\varepsilon)}(s))dV^{(\varepsilon)}(s)$$

General fast-slow systems III

Fortunately, rough path theory **works the same** for paths taking values in a **Banach space**.

We apply the same strategy - find a weak limit for the triple $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}, V^{(\varepsilon)})$ where

$$\mathbb{W}^{(\varepsilon)} = \varepsilon^{-2} \int_0^t \int_0^s h(\cdot, Y^{(\varepsilon)}(u)) \otimes h(\cdot, Y^{(\varepsilon)}(s)) duds \; .$$

This can be achieved by a (fairly) standard tightness + f.d.d. argument.

General fast-slow systems IV

By the continuity of the solution map, we obtain $X^{(\varepsilon)} \Rightarrow X$ where

$$X(t) = X(0) + \int_0^t H(X(s))d\mathbf{W}(s) + \int_0^t H(X(s))dV(s)$$

where $\mathbf{W} = (W, \mathbb{W})$ is an infinite dimensional "Brownian rough path" and $V(t) = \int f(\cdot, y)d\mu(y)t$.

This is a bit of a mess, but we can obtain a simpler formula by writing down the martingale problem.

General fast-slow systems V

Theorem (K. & Melbourne)

If the fast dynamics are "sufficiently chaotic" then $X^{(\varepsilon)} \Rightarrow X$ *where*

$$dX = \sigma(X)dB + \tilde{a}(X)dt\ ,$$

where B *is a standard BM on* $\mathbb{R}^d$ *and*

$$\tilde{a}(x) = \int f(x,y)d\mu(y) + \sum_{k=1}^{d} \mathfrak{B}(h^k(x,\cdot), \partial_k h(x,\cdot))$$

$$\sigma\sigma^T(x) = \mathfrak{B}(h^i(x,\cdot), h^j(x,\cdot)) + \mathfrak{B}(h^j(x,\cdot), h^i(x,\cdot))$$

and $\mathfrak{B}$ *is the "integrated autocorrelation" of the fast dynamics*

$$\mathfrak{B}(v,w) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu v(Y(0))v(Y(s))ds$$

The same idea even works for **discrete time** fast-slow systems.

Discrete time fast-slow systems

Suppose that $T : \Lambda \to \Lambda$ is a chaotic map with invariant measure μ. We consider the discrete fast-slow system

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2} h(X_j^{(n)}, T^j) + n^{-1} f(X_j^{(n)}, T^j)$$

Now define the path $X^{(n)}(t) = X_{\lfloor nt \rfloor}^{(n)}$.

The aim is to characterize the distribution of the path $X^{(n)}$ as $n \to \infty$.

Discrete time fast-slow systems

Akin to the continuous time picture, the limiting SDE can be determined by the limit of the pair $(W^{(n)}, \mathbb{W}^{(n)})$ where

$$W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(T^j)$$

and

$$\mathbb{W}^{(n),\alpha\beta}(t) = n^{-1} \sum_{0 \le i < j < \lfloor nt \rfloor} v^\alpha(T^i) v^\beta(T^j)$$

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. To appear in **Ann. Probab.** arXiv (2014).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2014).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. arXiv (2014).

All my slides are on my website (www.dtbkelly.com) **Thank you**!