EnKF and Catastrophic filter divergence

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The filtering problem

We have a **deterministic model**

$$\frac{dv}{dt} = F(v) \quad \text{with } v_0 \sim N(m_0, C_0) \; .$$

We will denote $v(t) = \Psi_t(v_0)$. Think of this as **very high dimensional** and **nonlinear**.

We want to **estimate** $v_j = v(jh)$ for some $h > 0$ and $j = 0, 1, \ldots, J$ given the **observations**

$$y_j = Hv_j + \xi_j \quad \text{for } \xi_j \text{ iid } N(0, \Gamma).$$

We can write down the condtional density using **Bayes' formula** ...

But for high dimensional nonlinear systems it's horrible.

Alternatively, we can use **EnKF** to draw **approximate samples** from the posterior.

The set-up for EnKF

Suppose we are given the ensemble $\{u_j^{(1)}, \dots, u_j^{(K)}\}$ at time j. For each particle, we create an **artificial observation**

$$y_{j+1}^{(k)} = y_{j+1} + \xi_{j+1}^{(k)} \quad , \quad \xi_{j+1}^{(k)} \text{ iid } N(0,\Gamma).$$

We update each particle using the **Kalman update**

$$u_{j+1}^{(k)} = \Psi_h(u_j^{(k)}) + G(u_j)\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right) \ ,$$

where $G(u_j)$ is the **Kalman gain** computed using the **forecasted ensemble covariance**

$$\widehat{C}_{j+1} = \frac{1}{K}\sum_{k=1}^{K}(\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)})^T(\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)}) \ .$$

There aren't many **theorems** about EnKF.

Ideally, we would like a theorem about **long time behaviour** of the filter for a finite ensemble size.

Filter divergence

It has been observed ($\star$) that when observations are **very frequent** the ensemble can **blow-up** (ie. reach machine-infinity) in **finite time**, even when the model has nice bounded solutions.

This is known as **catastrophic filter divergence**.

It is suggested in ($\star$) that this is caused by numerically integrating a stiff-system. Our aim is to "prove" this.

$\star$ Harlim, Majda (2010), Gottwald (2011), Gottwald, Majda (2013).

Assumptions (†)

1 - We make a **dissipativity** assumption on F. Namely that

$$F(\cdot) = A \cdot + B(\cdot, \cdot)$$

with A linear elliptic and B bilinear, satisfying certain estimates and symmetries.

Eg. 2d-Navier-Stokes, Lorenz-63, Lorenz-96.

2 - The **observation operator** $H = Id$ and the **noise covariance** $\Gamma = \gamma Id$

Discrete time results

For a fixed observation frequency $h > 0$ we can prove

Theorem (AS,DK)

If ($\dagger$) *then there exists constant* β *such that*

$$\mathbf{E}|u_j^{(k)}|^2 \leq e^{2\beta jh}\mathbf{E}|u_0^{(k)}|^2 + 2K\gamma^2 \left(\frac{e^{2\beta jh}-1}{e^{2\beta h}-1}\right)$$

Rmk. This becomes useless as $h \to 0$

For observations with $h \ll 1$, we need another approach.

The EnKF equations look like a discretization

Recall the ensemble update equation

$$\begin{aligned} u_{j+1}^{(k)} &= \Psi_h(u_j^{(k)}) + G(u_j)\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right) \\ &= \Psi_h(u_j^{(k)}) + \widehat{C}_{j+1}H^T(H^T\widehat{C}_{j+1}H + \Gamma)^{-1}\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right) \end{aligned}$$

Subtract $u_j^{(k)}$ from both sides and divide by h

$$\begin{aligned} \frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} &= \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} \\ &\quad + \widehat{C}_{j+1}H^T(hH^T\widehat{C}_{j+1}H + h\Gamma)^{-1}\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right) \end{aligned}$$

Clearly we need to rescale the noise (ie. Γ).

Continuous-time limit

If we set $\Gamma = h^{-1}\Gamma_0$ and substitute $y_{j+1}^{(k)}$, we obtain

$$\frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} = \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} + \widehat{C}_{j+1} H^T (h H^T \widehat{C}_{j+1} H + \Gamma_0)^{-1}$$
$$\left(H v + h^{-1/2}\Gamma_0^{1/2}\xi_{j+1} + h^{-1/2}\Gamma_0^{1/2}\xi_{j+1}^{(k)} - H\Psi_h(u_j^{(k)}) \right)$$

But we know that

$$\Psi_h(u_j^{(k)}) = u_j^{(k)} + O(h)$$

and

$$\begin{aligned}\widehat{C}_{j+1} &= \frac{1}{K}\sum_{k=1}^{K} (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)})^T (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)}) \\ &= \frac{1}{K}\sum_{k=1}^{K} (u_j^{(k)} - \overline{u_j})^T (u_j^{(k)} - \overline{u_j}) + O(h) = C(u_j) + O(h)\end{aligned}$$

Continuous-time limit

We end up with

$$\frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} = \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} - C(u_j)H^T\Gamma_0^{-1}H(u_j^{(k)} - v_j) + C(u_j)H^T\Gamma_0^{-1}\left(h^{-1/2}\xi_{j+1} + h^{-1/2}\xi_{j+1}^{(k)}\right) + O(h)$$

This looks like a **numerical scheme** for

$$\frac{du^{(k)}}{dt} = F(u^{(k)}) - C(u)H^T\Gamma_0^{-1}H(u^{(k)} - v) + C(u)H^T\Gamma_0^{-1/2}\left(\frac{dW^{(k)}}{dt} + \frac{dB}{dt}\right) . \qquad (\bullet)$$

Properties of the limiting equation

$$\frac{du^{(k)}}{dt} = F(u^{(k)}) - C(u)H^T\Gamma_0^{-1}H(u^{(k)} - v) \qquad (\bullet)$$
$$+ C(u)H^T\Gamma_0^{-1/2}\left(\frac{dW^{(k)}}{dt} + \frac{dB}{dt}\right) .$$

1 - Extra dissipation term only sees **differences in observed space**

2 - Extra dissipation only occurs in the **space spanned by ensemble**

3 - If F were linear, the equation for the mean is the equation for the **classical Kalman filter**, with an added sampling error.

4 - This is a **McKean-Vlasov** SDE.

Continuous-time results

Theorem (AS,DK)

Suppose the framework satisfies $(\dagger)$ *and* $\{u^{(k)}\}_{k=1}^{K}$ *satisfy* $(\bullet)$*. Let*

$$e^{(k)} = u^{(k)} - v \ .$$

Then there exists constant β *such that*

$$\mathbf{E}\sum_{k=1}^{K} |e^{(k)}(t)|^2 \leq \left(\mathbf{E}\sum_{k=1}^{K} |e^{(k)}(0)|^2\right) \exp\left(\beta t\right) \ .$$

Summary + Future Work

(1) Writing down an SDE/SPDE allows us to see the **important quantities** in the algorithm.

(2) Does not "prove" that catastrophic filter divergence is a numerical phenomenon, but is a decent starting point.

(1) Improve the condition on H.

(2) If we can **measure** the important quantities, then we can test the performance during the algorithm.

(3) Suggests new EnKF-like algorithms, for instance by discretising the stochastic PDE in a more **numerically stable** way.