Data assimilation in high dimensions

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What is data assimilation?

Suppose u satisfies

$$\frac{du}{dt} = F(u)$$

with some **unknown** initial condition u_0. We are most interested in geophysical models, so think high dimensional, nonlinear, possibly stochastic.

Suppose we make *partial, noisy* observations at times $t = h, 2h, \ldots, nh, \ldots$

$$y_n = Hu_n + \xi_n$$

where H is a linear operator (think low rank projection), $u_n = u(nh)$, and $\xi_n \sim N(0, \Gamma)$ iid.

The aim of **data assimilation** is to say something about the conditional distribution of u_n given the observations $\{y_1, \ldots, y_n\}$

How does filtering work: (initialization)

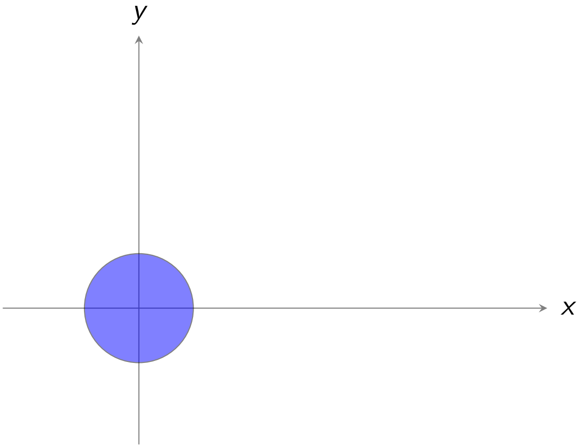


Figure: The blue circle represents our guess of u_0. Due to the uncertainty in u_0, this is a probability measure.

How does filtering work: (forecast)

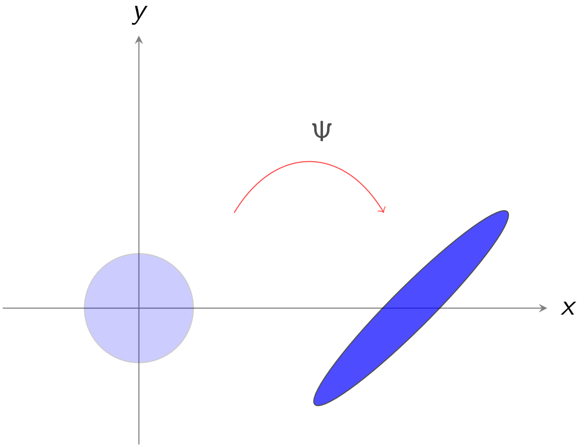


Figure: Apply the time h flow map Ψ. This produces a new probability measure which is our forecasted estimate of u_1. This is called the forecast step.

How does filtering work: (make an observation)

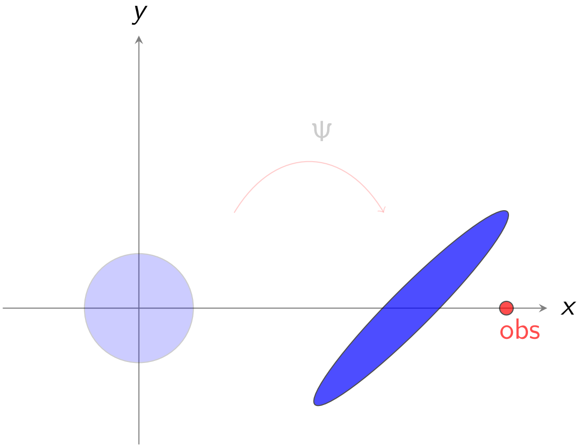


Figure: We make an observation $y_1 = Hu_1 + \xi_1$. In the picture, we only observe the x variable.

How does filtering work: (find best fit using Bayes)

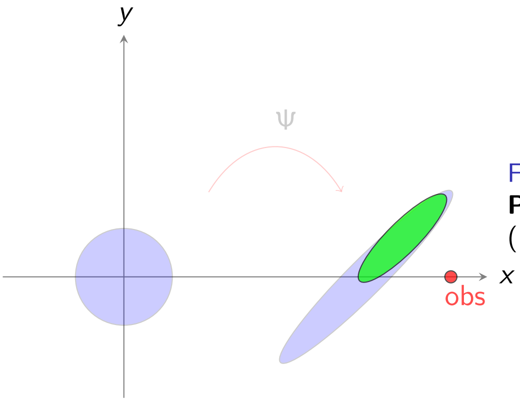


Figure:
$\mathbf{P}(u_1|y_1) \propto \mathbf{P}(y_1|u_1)\mathbf{P}(u_1)$
(Bayes formula)

Problems in high dimensions

In **numerical weather prediction** the state dimension is $O(10^9)$.

1) Difficult to store a density of this size

2) Computing the 'forecast step' is an integration over the state space.

We need **low dimensional** approximations of the filtering problem.

We will look at the **Ensemble Kalman filter**.

Ensemble Kalman filter (Evensen 94)

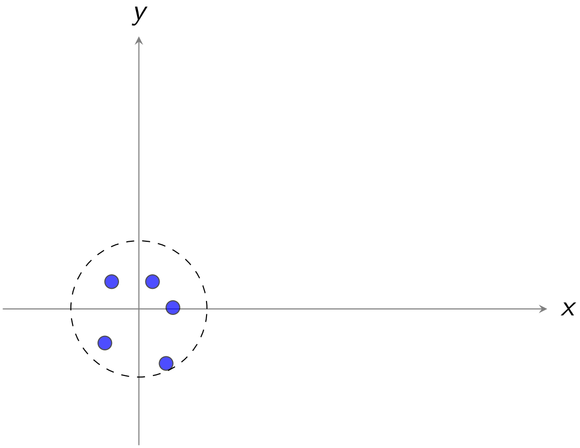


Figure: Start with K ensemble members drawn from some distribution. Empirical representation of u_0. The ensemble members are denoted $v_0^{(k)}$.

Only KN numbers are stored.

Ensemble Kalman filter (Forecast step)

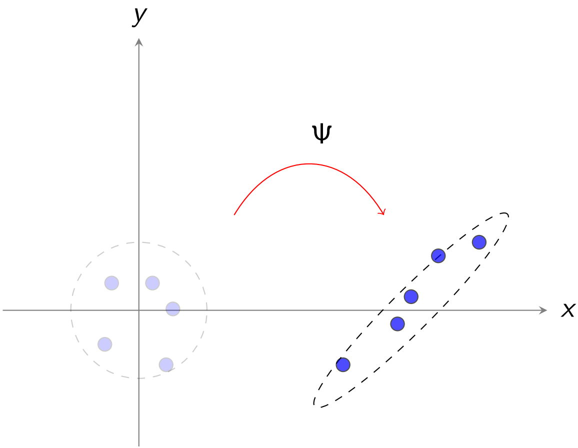


Figure: Apply the dynamics Ψ to each ensemble member.

$v_0^{(k)} \mapsto \Psi(v_0^{(k)})$

Ensemble Kalman filter (Make obs)

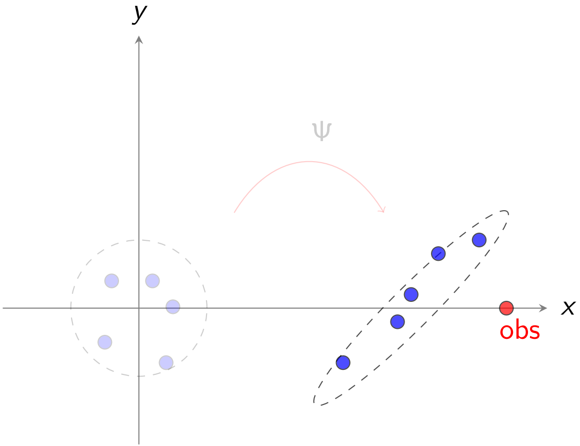


Figure: Make an observation.

Ensemble Kalman filter (Perturb obs)

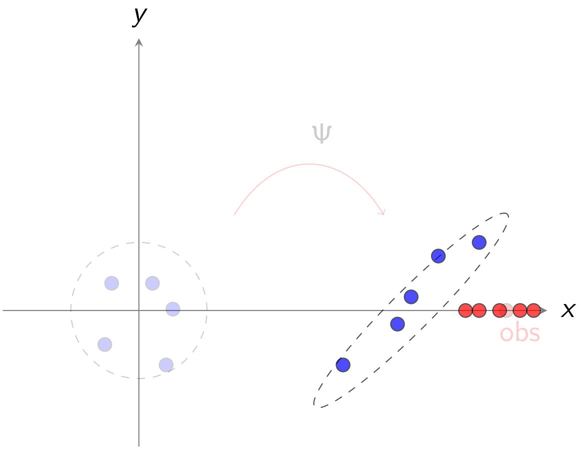


Figure: Turn the observation into K artificial observations by perturbing by the same source of observational noise.

$$y_1^{(k)} = y_1 + \xi_1^{(k)}$$

Ensemble Kalman filter (find best fit using Bayes)

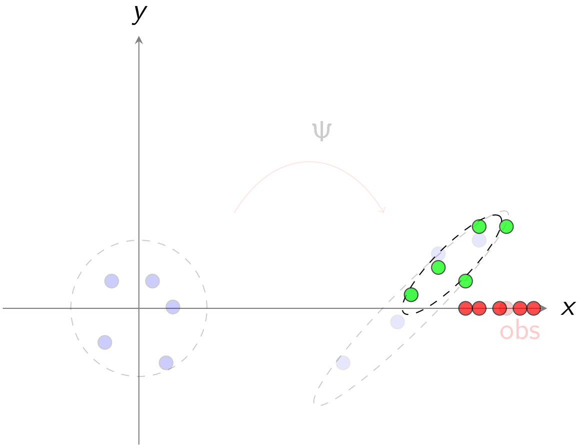


Figure: Update each member using the 'Kalman update formula'. This is a linear approximation of Bayes.

$$v_1^{(k)} = \Psi(v_0^{(k)}) + K_1(y_1^{(k)} - H\Psi(v_0^{(k)}))$$

Why should mathematicians be interested?

A widely used algorithm (NWP, disease forecasting, chemical reactions) with many questions and not so many answers:

1 - Is the filter stable to perturbations? *eg. Will different initializations converge? (ergodicity)*

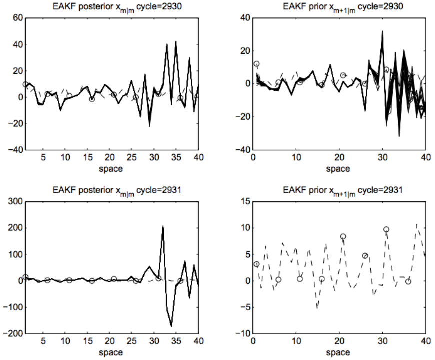
2 - Is the filter accurate? *Is the posterior consistent with the true signal?*

3 - Can we design mathematically sensible alternative algorithms?

4 - Can we understand why/when the filter fails?

Catastrophic filter divergence

Lorenz-96: $\dot{u}_j = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F$ with $j = 1, \ldots, 40$. Periodic BCs. Observe every fifth node. *(Harlim-Majda 10, Gottwald-Majda 12)*



True solution in a bounded set, but filter **blows up** to machine infinity in finite time!

For complicated models, only heuristic arguments offered as explanation.

Can we **prove** *it for a simpler constructive model?*

The rotate-and-lock map (K., Majda, Tong. PNAS 15.)

The model $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of two maps $\Psi(x, y) = \Psi_{lock}(\Psi_{rot}(x, y))$ where

$$\Psi_{rot}(x, y) = \begin{pmatrix} \rho\cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and Ψ_{lock} rounds the input to the nearest point in the grid

$$\mathcal{G} = \{(m, (2n+1)\varepsilon) \in \mathbb{R}^2 : m, n \in \mathbb{Z}\} .$$

It is easy to show that this model has an **energy dissipation principle**:

$$|\Psi(x, y)|^2 \le \alpha|(x, y)|^2 + \beta$$

for $\alpha \in (0, 1)$ and $\beta > 0$.

(a)

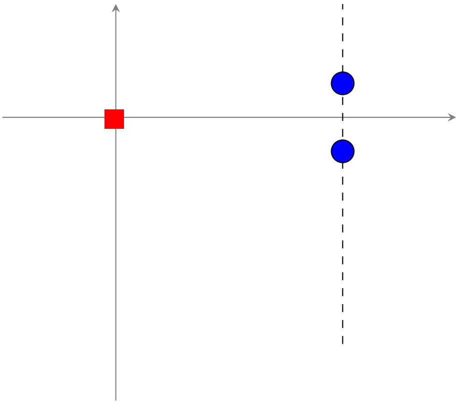


Figure: The red square is the trajectory $u_n = 0$. The blue dots are the positions of the forecast ensemble $\Psi(v_0^+)$, $\Psi(v_0^-)$. Given the locking mechanism in Ψ, this is a natural configuration.

(b)

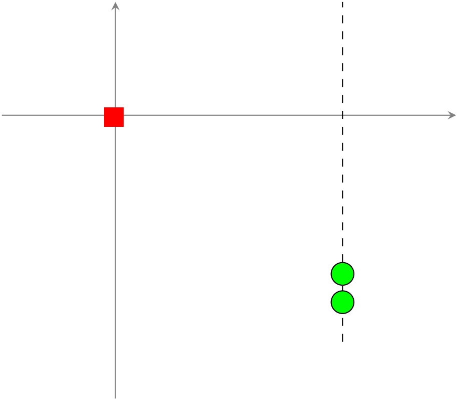


Figure: We make an observation (H shown below) and perform the analysis step. The green dots are v_1^+, v_1^-.

Observation matrix
$$H = \begin{pmatrix} 1 & 0 \\ \varepsilon^{-2} & 1 \end{pmatrix}$$

The filter is 'sure' that $u_1 = \hat{x}$ (the dashed line). The filter deduces that the observation is approximately $(y_1, y_2) = (\hat{x}, \varepsilon^{-2}\hat{x} + u_2)$.

Thus $v_1^{\pm} \approx (\hat{x}, -\varepsilon^{-2}\hat{x})$

(c)

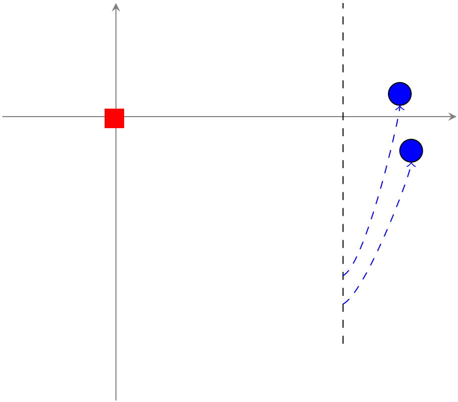


Figure: Beginning the next assimilation step. Apply Ψ_{rot} to the ensemble (blue dots)

(d)

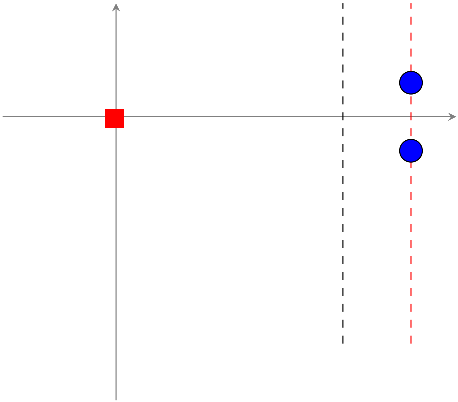


Figure: Apply Ψ_{lock}. The blue dots are the forecast ensemble $\Psi(v_1^+)$, $\Psi(v_1^-)$. Exact same as frame 1, but higher energy orbit. The cycle repeats leading to **exponential growth**.

Theorem (K.-Majda-Tong 15 PNAS)

For any $N > 0$ and any $p \in (0,1)$ there exists a choice of parameters such that

$$\mathbf{P}\left(|v_n^{(k)}| \geq M_n \ \textit{for all } n \leq N\right) \geq 1 - p$$

where M_n is an exponentially growing sequence.

ie - The filter can be made to grow exponentially for an arbitrarily long time with an arbitrarily high probability.

References

1 - D. Kelly, K. Law & A. Stuart. *Well-Posedness And Accuracy Of The Ensemble Kalman Filter In Discrete And Continuous Time.* **Nonlinearity** (2014).

2 - D. Kelly, A. Majda & X. Tong. *Concrete ensemble Kalman filters with rigorous catastrophic filter divergence*. **Proc. Nat. Acad. Sci.** (2015).

3 - X. Tong, A. Majda & D. Kelly. *Nonlinear stability and ergodicity of ensemble based Kalman filters*. **Nonlinearity** (2016).

4 - X. Tong, A. Majda & D. Kelly. *Nonlinear stability of the ensemble Kalman filter with adaptive covariance inflation*. To appear in **Comm. Math. Sci.** (2015).

All my slides are on my website (www.dtbkelly.com) **Thank you**!