Stochastic Modelling in Climate Science

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Why use stochastic models?

The basic system we are trying to model is of the form

$$\frac{dx}{dt} = F(x, y)$$

where x are **resolved** variables evolving on a slow timescale and y are **unresolved** variables evolving on a fast timescale.

Eg. x are climate variables, with a response time of **years** and y are weather effects, with a response time of **hours**.

Because of this structure, these systems exhibit features of **stochastic processes** - most importantly **variability**.

Outline

1 - Building a **stochastic model** - SDEs.

2 - **Stochastic calculus** ... different to normal calculus

3 - **Statistics** of SDEs

4 - **Numerical schemes** for SDEs.

1.

How can we **build** a **stochastic model**?

Building a stochastic model

Suppose we are trying to model a perturbed system

$$\frac{dx}{dt} = F(x) + \text{noise}$$

We build this model using an **approximation**.

Fix some $\Delta t \ll 1$ and let $x_k \approx x(k\Delta t)$. If the noise is independent of x, then we can write

$$x_{k+1} = x_k + F(x_k)\Delta t + \Delta W_k \ .$$

Think of ΔW_k as all the noise **accumulated** over the time step Δt.

What properties should we require of ΔW_k?

There are a few natural assumptions to make about ΔW_k that make the model a lot simpler.

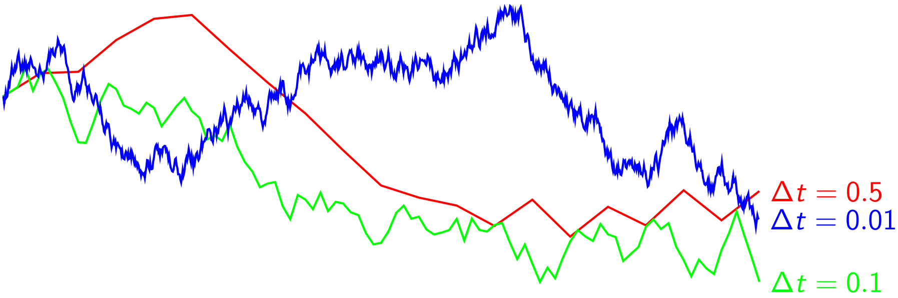
1. The sequence $\Delta W_1, \Delta W_2, \Delta W_3, \ldots$ should be **i.i.d**.
2. ΔW_k should be **Gaussian**.
3. $\mathbf{E}\Delta W_k = 0$.
4. $\mathbf{E}\Delta W_k^2 \sim \Delta t$.

Brownian motion

Since ΔW_k are **noise increments**, we should add them up!

$$W(t) \approx \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} \Delta W_k$$

In the limit $\Delta t \to 0$, the random path is called **Brownian motion**.



Building a stochastic model

Returning to the approximate model

$$x_{k+1} = x_k + F(x_k)\Delta t + \Delta W_k \ .$$

To see what “ODE” this represents, we write

$$\frac{x_{k+1} - x_k}{\Delta t} = F(x_k) + \frac{\Delta W_k}{\Delta t} \ ,$$

this is clearly an approximation of

$$\frac{dx}{dt} = F(x) + \frac{dW}{dt} \ .$$

The object $\frac{dW}{dt}$ is called **white noise**.

Building a stochastic model

As an ODE, the model is not particularly well defined, since W is **nowhere differentiable**. That means $\frac{dW}{dt}$ is nowhere defined!

This is not surprising, since

$$\mathbf{E}\left(\frac{\Delta W_k}{\Delta t}\right)^2 \sim \frac{1}{\Delta t} \to \infty \,.$$

Building a stochastic model

Mathematically, it doesn't matter that the ODE is not well defined. The integral equation is well defined

$$x_{k+1} = x_k + F(x_k)\Delta t + \Delta W_k \ .$$

Then $x(t) = x_{\lfloor t/\Delta t \rfloor}$ is given by

$$x(t) = x(0) + \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} F(x_k)\Delta t + \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} \Delta W_k$$

This is clearly an approximation of

$$x(t) = x(0) + \int_0^t F(x(s))ds + W(t)$$

Building a stochastic model

The equation

$$x(t) = x(0) + \int_0^t F(x(s))ds + W(t)$$

is called a **Stochastic Differential Equation** (**SDE**).

We often use the shorthand

$$dx = F(x)dt + dW$$

When the noise doesn't depend on the solution x, the noise is called **additive**.

Building a stochastic model: multiplicative noise

Suppose the **magnitude** of the noise depends on the **state** of the model

$$x(t) = x(0) + \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} F(x_k)\Delta t + \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} G(x_k)\Delta W_k$$

Under certain assumptions on $G(x)$, the limit of $\sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} G(x_k)\Delta W_k$ exists and is called an **Itô integral**.

The limit becomes

$$x(t) = x(0) + \int_0^t F(x(s))ds + \int_0^t G(x(s))dW(s) \ .$$

In shorthand, this is written

$$dx = F(x)dt + G(x)dW \ .$$

Stochastic Differential Equations

There are several different interpretations as to what it means to be a solution to the SDE

$$dx = F(x)dt + G(x)dW \ .$$

To an applied mathematician, the most natural is simply that x is the limit of the approximation defined in the previous slides.

A more rigorous way is to define the Itô integral $\int YdW$ for some space of random paths Y, and then construct a fixed point argument on that space.

2.

How does **stochastic calculus** work?

It is **natural** to think that

$$dx = \frac{dx}{dt}dt$$

But for SDEs this is **false**... If x isn't differentiable, then **normal calculus** doesn't work.

Stochastic calculus

Eg. Suppose we want to write down an SDE whose solution is $x(t) = W^2(t)$. One would expect that

$$dx = 2W\,dW$$

but this is wrong! To see why, we go back to the discretization

$$\begin{aligned} x_{k+1} - x_k &= W_{k+1}^2 - W_k^2 = (W_{k+1} + W_k)(W_{k+1} - W_k) \\ &= 2W_k(W_{k+1} - W_k) + (W_{k+1} - W_k)(W_{k+1} - W_k) \\ &= 2W_k \Delta W_k + (\Delta W_k)^2 \end{aligned}$$

Stochastic calculus

Adding them up

$$x(t) = x(0) + 2 \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} W_k \Delta W_k + \sum_{k=0}^{\lfloor t/\Delta t \rfloor - 1} (\Delta W_k)^2$$

The first sum (by definition) converges to an Itô integral. The limit of the second sum can be computed using the **Law of Large Numbers** (like the ergodic theorem). We obtain the limit

$$x(t) = x(0) + 2 \int_0^t W(s) dW(s) + t \; .$$

Or in short

$$dx = 2W dW + dt$$

Itô's formula

In general, the rules of **stochastic calculus** is determined by **Itô's formula**. This is a **stochastic chain-rule**.

Theorem

Suppose that x *is the solution to*

$$dx = F(x)dt + G(x)dW$$

and that ϕ *is some smooth enough function. Then*

$$\begin{aligned} d\phi(x) &= \phi'(x)dx + \frac{1}{2}\phi''(x)G^2(x)dt \\ &= \phi'(x)(F(x)dt + G(x)dW) + \frac{1}{2}\phi''(x)G^2(x)dt \end{aligned}$$

An example of Itô's formula

Consider the following stochastic model called **geometric Brownian motion** (gBm) (stock price, population model with noisy growth rate)

$$dx = rx\,dt + \sigma x\,dW \;,$$

where r, σ are constants.

To solve this using **normal** calculus, we would write

$$\frac{dx}{x} = r\,dt + \sigma\,dW$$

then integrate. Instead we must use **Itô's formula**.

An example of Itô's formula

By Itô's formula we have

$$d\log(x) = \frac{dx}{x} - \frac{1}{2x^2}(\sigma x)^2 dt = (r - \frac{1}{2}\sigma^2)dt + \sigma dW \ .$$

And integrating, we get

$$\log(x(t)) = \log(x(0)) + (r - \frac{1}{2}\sigma^2)t + \sigma W(t)$$

so

$$x(t) = x(0)\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right)$$

Stratonovich integrals

Stochastic models are very **sensitive** to the **source of noise**. Suppose that $W^\varepsilon \to W$ was a **smooth approximation** of Brownian motion. Then the (random) ODE makes perfect sense.

$$\frac{dx^\varepsilon}{dt} = F(x^\varepsilon) + G(x^\varepsilon)\frac{dW^\varepsilon}{dt}$$

We can define the stochastic model as the limit $x^\varepsilon \to x$ as $\varepsilon \to 0$.

One would guess that x solves

$$dx = F(x)dt + G(x)dW$$

But it **doesn't**!

Stratonovich integrals

Eg. Back to the gBm example, suppose that

$$\frac{dx^{\varepsilon}}{dt} = rx^{\varepsilon} + \sigma x^{\varepsilon} \frac{dW^{\varepsilon}}{dt} \ .$$

We will show that the limit is **not**

$$dx = rxdt + \sigma x dW$$

For each fixed ε, since everything is piecewise smooth, normal calculus works. So in fact

$$\frac{d}{dt} \log(x^{\varepsilon}) = r + \sigma \frac{dW^{\varepsilon}}{dt}$$

and

$$x^{\varepsilon}(t) = x(0) \exp\left(rt + W^{\varepsilon}(t)\right)$$

Stratonovich integrals

The limit is clearly

$$x(t) = x(0)\exp\left(rt + W(t)\right) \ .$$

One can check that this solves the SDE

$$dx = (r + \frac{1}{2}\sigma^2)x dt + \sigma x dW \ .$$

When the noise arises in this way, one instead writes

$$dx = rx dt + \sigma x \circ dW \ ,$$

and the stochastic integral is called a **Stratonovich integral**. It is easy to convert between Itô and Stratonovich integrals.

Itô vs Stratonovich

From a modeling standpoint, one should decide **a priori** how their noise enters the model.

If the noise enters as a **discrete process** (e.g. weather effects like rainfall) then one should use **Itô integrals**.

If the noise enters as a **continuous process** (e.g. fast chaotic effects) then one should use **Stratonovich integrals**.

Recap

We have seen that

1 - SDEs arise naturally as stochastic models.

2 - SDEs have their own calculus.

3 - SDEs are sensitive to the source of noise.

3.

Main advantage of SDEs - their **statistics** are extremely well understood.

The statistics of SDEs

The statistical properties of SDEs are very well understood. Let's look at our *gBm* example.

$$x(t) = x(0) + r\int_0^t x(s)ds + \sigma\int_0^t x(s)dW(s)\ .$$

We can compute the **mean**. Clearly we have

$$\mathbf{E}x(t) = \mathbf{E}x(0) + r\int_0^t \mathbf{E}x(s)ds + \sigma\mathbf{E}\left(\int_0^t x(s)dW(s)\right)\ .$$

But $\mathbf{E}\left(\int_0^t x(s)dW(s)\right) = 0$. Why? Look at the discretization again

$$\mathbf{E}\left(\int_0^t x(s)dW(s)\right) = \mathbf{E}\left(\sum_{k=0}^{\lfloor t/\Delta t\rfloor -1} x_k \Delta W_k\right) = \sum_{k=0}^{\lfloor t/\Delta t\rfloor -1} \mathbf{E}x_k\mathbf{E}\Delta W_k = 0$$

This follows from the fact that x_k only depends on the **past** increments of ΔW and must be **independent** of ΔW_k.

The statistics of SDEs

Hence we have

$$\mathbf{E}x(t) = x(0) + \int_0^t r\mathbf{E}x(s)ds \ .$$

In particular, the mean follows the original noise-free model. This is true for all SDEs where the noise-free model is linear.

If $m(t) = \mathbf{E}x(t)$ then

$$\frac{dm}{dt} = rm \ .$$

The statistics of SDEs

We can compute the **variance** using Itô's formula

$$d(x^2) = 2x\,dx + \sigma^2 x^2 dt = 2(r+\sigma^2)x^2 dt + 2\sigma x^2 dW$$

Hence

$$\mathbf{E}x(t)^2 = \mathbf{E}x(0)^2 + 2(r+\sigma^2)\int_0^t \mathbf{E}x(s)^2 ds$$

and so if $v(t) = \mathbf{E}(x(t) - m(t))^2$ then

$$\frac{dv}{dt} = 2\sigma^2 v \; .$$

Connection to PDEs

For all other **statistics**, there is an extremely useful tool. The **density** $\rho(z,t)$ of $x(t)$ is the solution to the PDE

$$\partial_t \rho(z,t) = -\partial_z (F(z)\rho(z,t)) + \frac{1}{2}\partial_z^2 (G(z)^2 \rho(z,t)) \;,$$

where the initial condition is the density of $x(0)$, for instance a Dirac delta.

This is called the **Fokker-Planck equation**.

Connection to PDEs

Eg. Suppose we want the density of Brownian motion ($dx = dW$) started from $W(0) = 0$. Then

$$\partial_t \rho(z,t) = \frac{1}{2}\partial_z^2 \rho(z,t)\ .$$

This is just the **heat equation**.If initial condition is δ_0 then

$$\rho(z,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-z^2}{2t}\right)$$

Unsurprising given that $W(t)$ must be a Gaussian with $\mathbf{E}W(t) = 0$ and $\mathbf{E}W(t)^2 = t$.

4.

Numerical schemes for SDEs.

Numerics for SDEs

Alternatively, one can generate statistics by **sampling**. This can be achieved by solving the equations **numerically**.

Numerics are extremely important - Most **practitioners** only interact with the numerics.

Unfortunately, they are quite **subtle**. As we saw from the Stratonovich example, it is common for a natural approximation to yield **wrong** SDE in the limit.

Numerics for SDEs

The most common scheme is the one used to introduce SDEs. Namely,

$$x_{k+1} = x_k + F(x_k)\Delta t + G(x_k)\Delta W_k \ .$$

One obtains ΔW_k by simulating a Gaussian random variable (which is easy). This is called the **Euler-Maruyama** scheme.

It always yields the **correct** limit.

As with ODEs, the EM scheme is **not very stable**. For **non-linear systems**, you might have to use extremely small Δt to get a reasonable answer.

Numerics for SDEs

A traditional ODE way around this stability problem is to make the scheme **implicit**. For example, the **trapezoidal rule**

$$x_{k+1} = x_k + \frac{F(x_k) + F(x_{k+1})}{2} \Delta t + \frac{G(x_k) + G(x_{k+1})}{2} \Delta W_k \ .$$

But this gives the wrong limit! In fact the limit of this scheme is

$$dx = F(x)dt + G(x) \circ dW \ .$$

The same SDE, but with Stratonovich instead of Itô.

Moral: be careful with your numerics ...

Next lecture ...

Looking at **REAL** stochastic climate models.